Integration By Parts Solutions:

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1.1 Parts Once

1.1.1 Power and Trigonometry

1) $\int x \sin 2x \, dx$

When the integral is a product of two completely unrelated functions (and we can't use reverse chain rule or substitution) we use parts. Such as when we have:

- $\int$ Power × Trig — e.g. $\int x \sin x \, dx$ or $\int x^2 \sin x \, dx$
- $\int$ Power × Exponential — e.g. $\int x e^x dx$ or $\int x^2 e^x dx$
- $\int$ Easy power × Harder power — e.g. $\int 5x\sqrt{2-x} \, dx$
- $\int$ Power × Ln — e.g. $\int x \ln x \, dx$ or $\int x^2 \ln x \, dx$ or $\int \frac{1}{x} \ln x dx$
- $\int$ Exponential × Trig — e.g. $\int e^x \sin x \, dx$ or $\int e^x \cos x \, dx$

Step 1:

We say our integral has the form/looks like $\int u \frac{dv}{dx} dx$ or $\int \frac{dv}{dx} u \, dx$ when we want to use parts. This means when we do parts, we look at the functions inside the integral and we call one of the functions u and the other function - $\frac{dv}{dx}$. It matters which we call u and which we call $\frac{dv}{dx}$!

To choose which function is u and which is $\frac{dv}{dx}$ we can use the following criteria **LIATE**

Logarithms	$\ln x, \log x$
Inverse Trig	$\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$
Algebra	constants, $x, x^2, \ldots$
Trigonometry	$\sin x, \cos x, \tan x$
Exponentials	e^x

Locate the two functions given inside in the integral. Whichever function out of the 2 functions comes first on the list above is always called u.

In this question we have

$$\int x \sin 2x \, dx$$

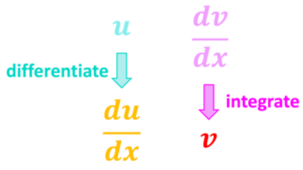
x is the Algebra function

$\sin 2x$ is the Trig function

Algebra Function (A) comes before the Trig function (T) on the LIATE list above so we call the Algebra function u and the Trig function $\frac{dv}{dx}$.

Step 2:

Find $\frac{du}{dx}$ by differentiating u and v by integrating $\frac{dv}{dx}$



So, in our case we get

$$\int x \sin 2x \, dx$$

$$u = x \qquad \frac{dv}{dx} = \sin 2x$$

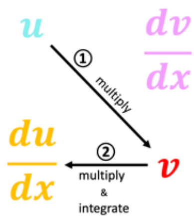
$$\frac{du}{dx} = 1 \qquad v = \frac{-\cos 2x}{2} = -\frac{1}{2}\cos 2x$$

Notice how for both functions we want to end up with the original function and its corresponding derivative which leads us to step 3.

Step 3:

Plug into the formula $uv - \int v \frac{du}{dx} dx$

Some students struggle to remember this formula. It can help to remember the formula as ① – ② in the pattern below:



So, plugging into $uv - \int v \frac{du}{dx} dx$ gives

$$= x\left(-\frac{1}{2}\cos 2x\right) - \int \left(-\frac{1}{2}\cos 2x\right)(1)dx$$

Step 4:

Simplify the integral part $\int v \frac{du}{dx} dx$ as much as possible before integrating. You take out constants (that are multiplied or divided) to make the integration easier.

Simplifying gives

$$= -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x \, dx$$

Taking out the constant $-\frac{1}{2}$ to make it easier:

$$= -\frac{1}{2}x\cos 2x - \left(-\frac{1}{2}\right)\int \cos 2x \, dx$$

Integrating (always remember the constant of integration + c)

$$= -\frac{1}{2}x\cos 2x + \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) + c$$

Simplifying:

$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$$

2) $\int x \cos\frac{x}{2}\, dx$

We will skip the detailed steps since they have been shown clearly in question 1.

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= x\left(2\sin\frac{x}{2}\right) - \int\left(2\sin\frac{x}{2}\right)(1)\, dx$$

Simplify:

$$= 2x\sin\frac{x}{2} - \int 2\sin\frac{x}{2} dx$$

Taking out the constant 2 to make it easier:

$$= 2x\sin\frac{x}{2} - 2\int \sin\frac{x}{2} dx$$

Integrating:

$$= 2x\sin\frac{x}{2} - 2\left(\frac{-\cos\frac{x}{2}}{\frac{1}{2}}\right) + c$$

Simplify:

$$= 2x\sin\frac{x}{2} + 4\cos\frac{x}{2} + c$$

(Always remember the constant of integration + c)

3) $\int_0^{\frac{\pi}{2}} x\sin 3x dx$

$$\int_0^{\frac{\pi}{2}} x\sin 3x\; dx$$

$u = x$ $\qquad$ $\frac{dv}{dx} = \sin 3x$

↓ $\qquad$ ↓

$\frac{du}{dx} = 1$ $\qquad$ $v = \frac{-\cos 3x}{3} = -\frac{1}{3}\cos 3x$

Plug into the formula $uv - \int v \frac{du}{dx} dx$

$$= x\left(-\frac{\cos 3x}{3}\right) - \int\left(-\frac{1}{3}\cos 2x\right)(1)dx$$

Simplify what is inside the integral:

$$= -\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x\, dx$$

Take the constant $-\frac{1}{3}$ out of the integral to make the integral easier to integrate:

$$= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x \, dx$$

Integrate:

$$= -\frac{1}{3}x\cos 3x + \frac{1}{3}\left(\frac{\sin 3x}{3}\right)$$

(since we have limits, we don't need to add the constant of integration + c)

Simplify:

$$= -\frac{1}{3}x\cos 3x + \frac{\sin 3x}{9}$$

Apply the limits:

$$= \left[-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x\right]_0^{\frac{\pi}{2}}$$

$$= \left(-\frac{\pi}{6}\cos\frac{3\pi}{2} + \frac{1}{9}\sin\frac{3\pi}{2}\right) - (0+0))$$

$$= -\frac{\pi}{6}(0) + \frac{1}{9}(-1)$$

$$= -\frac{1}{9}$$

Note: We could have applied the limits as we were going, but it was easier to do it at the end once all the integration was done

1.1.2 Power and Exponential

4) $\int xe^{4x}dx$

$$\int xe^{4x}dx$$

$$u = x \qquad \frac{dv}{dx} = e^{4x}$$

$$\downarrow \qquad \downarrow$$

$$\frac{du}{dx} = 1 \qquad v = \frac{1}{4}e^{4x}$$

Plugging into formula: $uv - \int v\frac{du}{dx}dx$

$$= x\left(\frac{1}{4}e^{4x}\right) - \int\left(\frac{1}{4}e^{4x}\right)(1)dx$$

Simplify:

$$= \frac{1}{4}xe^{4x} - \int \frac{1}{4}e^{4x}\, dx$$

Taking out the constant $\frac{1}{4}$ to make it easier:

$$= \frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x}dx$$

Integrating:

$$= \frac{1}{4}xe^{4x} - \frac{1}{4}\left(\frac{1}{4}e^{4x}\right) + c$$

Simplify:

$$= \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + c$$

(Always remember the constant of integration + c)

5) $\int_0^1 xe^{2x}dx$

$$\int_0^1 xe^{2x}dx$$

$u = x \quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 1 \quad v = \frac{1}{2}e^{2x}$

Plugging into formula: $uv - \int v\frac{du}{dx}dx$

$$= x\left(\frac{1}{2}e^{2x}\right) - \int\left(\frac{1}{2}e^{2x}\right)(1)dx$$

Simplify:

$$= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\ dx$$

Taking out the constant $\frac{1}{2}$ to make it easier:

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\ dx$$

Integrating:

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right)$$

Simplify:

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_0^1$$

$$= \left(\frac{1}{2}e^2 - \frac{1}{4}e^2\right) - \left(0 - \frac{1}{4}e^0\right)$$

$$= \left(\frac{1}{4}e^2\right) - \left(-\frac{1}{4}\right)$$

$$=\frac{1}{4}e^2+\frac{1}{4}$$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

1.1.3 Power and Ln

6) $\int x \ln 2x dx$

$$\int x \ln 2x \quad dx$$

$u = \ln 2x$ → $\frac{du}{dx} = \frac{2}{2x} = \frac{1}{x}$

$\frac{dv}{dx} = x$ → $v = \frac{x^2}{2} = \frac{1}{2}x^2$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= \ln 2x \left(\frac{1}{2}x^2\right) - \int \left(\frac{1}{2}x^2\right)\left(\frac{1}{x}\right) dx$$

Simplify:

$$= \frac{1}{2}x^2 \ln 2x - \int \frac{1}{2} x \, dx$$

Taking out the constant $\frac{1}{2}$ to make it easier:

$$= \frac{1}{2}x^2 \ln 2x - \frac{1}{2}\int x \, dx$$

Integrating:

$$= \frac{1}{2}x^2 \ln 2x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

Simplify:

$$= \frac{1}{2}x^2 \ln 2x - \frac{1}{4}x^2 + c$$

(Always remember the constant of integration + c)

7) $\int_1^e x^2 \ln x \, dx$

$$\int_1^e x^2 \ln x \; dx$$

$u = \ln x \qquad \frac{dv}{dx} = x^2$

$\downarrow \qquad \downarrow$

$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{x^3}{3} = \frac{1}{3}x^3$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= \ln x \left(\frac{1}{3}x^3\right) - \int \left(\frac{1}{3}x^3\right)\left(\frac{1}{x}\right) dx$$

Simplify:

$$= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2 dx$$

Taking out the constant $\frac{1}{3}$ to make it easier:

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$$

Integrating:

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\left(\frac{1}{3}x^3\right)$$

Simplify:

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= \left[\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3\right]_1^e$$

$$= \left(\frac{1}{3}e^3 \ln e - \frac{1}{9}e^3\right) - \left(\frac{1}{3}\ln 1 - \frac{1}{9}(1^3)\right)$$

$$= \frac{1}{3}e^3 - \frac{1}{9}e^3 - 0 + \frac{1}{9}$$

$$= \frac{2}{9}e^3 + \frac{1}{9}$$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

8) $\int_1^2 \frac{1}{x^3} \ln x \, dx$

$$\int_1^2 x^{-3} \ln x \, dx$$

$u = \ln x$ $\quad\quad$ $\frac{dv}{dx} = x^{-3}$

$\frac{du}{dx} = \frac{1}{x}$ $\quad\quad$ $v = \frac{x^{-2}}{-2} = -\frac{1}{2}x^{-2}$

Plugging into formula: $uv - \int v\frac{du}{dx}dx$

$$= \ln x\left(-\frac{1}{2}x^{-2}\right) - \int \left(-\frac{1}{2}x^{-2}\right)\left(\frac{1}{x}\right)dx$$

Simplify:

$$= -\frac{1}{2}x^{-2}\ln x - \int -\frac{1}{2}x^{-3}dx$$

Taking out the constant $-\frac{1}{2}$ **to make it easier:**

$$= -\frac{1}{2}x^{-2}\ln x + \frac{1}{2}\int x^{-3}dx$$

Integrating:

$$= -\frac{1}{2}x^{-2}\ln x + \frac{1}{2}\left(\frac{x^{-2}}{-2}\right)$$

Simplify:

$$= -\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2}$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= \left[-\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2}\right]_1^2$$

$$= \left(-\frac{1}{2}2^{-2}\ln 2 - \frac{1}{4}2^{-2}\right) - \left(-\frac{1}{2}\ln 1 - \frac{1}{4}(1^{-2})\right)$$

$$= \left(-\frac{1}{8}\ln 2 - \frac{1}{16}\right) - \left(0 - \frac{1}{4}\right)$$

$$= -\frac{1}{8}\ln 2 - \frac{1}{16} + \frac{1}{4}$$

$$= \frac{3}{16} - \frac{1}{8}\ln 2$$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

1.1.4 Power And Power

9) $\int x(x+2)^5 dx$

$$\int x\,(x+2)^5\,dx$$

$u = x$ $\quad\quad$ $\frac{dv}{dx} = (x+2)^5$

$\frac{du}{dx} = 1 \qquad v = \frac{(x+2)^6}{6}$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$= x\left(\frac{(x+2)^6}{6}\right) - \int \left(\frac{(x+2)^6}{6}\right)(1) dx$

Simplify:

$= \frac{x}{6}(x+2)^6 - \int \frac{(x+2)^6}{6} dx$

Taking out the constant $\frac{1}{6}$ to make it easier:

$= \frac{x}{6}(x+2)^6 - \frac{1}{6}\int (x+2)^6\, dx$

Integrating:

$= \frac{x}{6}(x+2)^6 - \frac{1}{6}\left(\frac{(x+2)^7}{7}\right) + c$

Simplify:

$= \frac{x}{6}(x+2)^6 - \frac{(x+2)^7}{42} + c$

(Always remember the constant of integration + c)

2 Silver

2.1 Parts Once

2.1.1 Power and Exponential

10) $\int_0^8 (4xe^{-\frac{1}{3}x} + 3)dx$. Find the exact value

$$\int_0^8 \left(4xe^{-\frac{1}{3}x} + 3\right) dx$$

$$= \int_0^8 4xe^{-\frac{1}{3}x} dx + \int_0^8 3\, dx$$

Taking out constant 4 from the first integral to make it easier:

$$= 4\int_0^8 xe^{-\frac{1}{3}x} dx + \int_0^8 3\, dx$$

$$= 4\int_0^8 xe^{-\frac{1}{3}x} dx + \int_0^8 3\, dx$$

$u = x \quad\rightarrow\quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-\frac{1}{3}x} \quad\rightarrow\quad v = \frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}} = -3e^{-\frac{1}{3}x}$

use parts

Doesn't require parts. we can integrate straight away

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= 4\left[x\left(-3e^{-\frac{1}{3}x}\right) - \int\left(-3e^{-\frac{1}{3}x}\right)(1)dx\right] + 3x$$

Simplify:

$$= 4\left[-3xe^{-\frac{1}{3}x} - \int -3e^{-\frac{1}{3}x}\, dx\right] + 3x$$

Taking out the constant -3 inside the integral to make it easier:

$$= 4\left[-3xe^{-\frac{1}{3}x} + 3\int e^{-\frac{1}{3}x}\,dx\right] + 3x$$

Integrating:

$$= 4\left[-3xe^{-\frac{1}{3}x} + 3\left(-3e^{-\frac{1}{3}x}\right)\right] + 3x$$

Simplify:

$$= -12xe^{-\frac{1}{3}x} - 36e^{-\frac{1}{3}x} + 3x$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= \left[-12xe^{-\frac{1}{3}x} - 36e^{-\frac{1}{3}x} + 3x\right]_0^8$$

$$= \left(-12(8)e^{-\frac{8}{3}} - 36e^{-\frac{8}{3}} + 3(8)\right) - (0 - 36e^0 + 3(0))$$

$$= (-96e^{-\frac{8}{3}} - 36e^{-\frac{8}{3}} + 25) - (-36)$$

$$= (-132e^{-\frac{8}{3}} + 24) + 36$$

$$= 60 - 132e^{-\frac{8}{3}}$$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

11) $\int_0^2 (2e^{2x} - xe^{2x})dx$. Show that this is $\frac{1}{4}e^4 - \frac{5}{4}$

$$\int_0^2 (2e^{2x} - xe^{2x})dx$$

$$= \int_0^2 2e^{2x}dx - \int_0^2 xe^{2x}\,dx$$

Taking out constant 2 from the first integral to make it easier:

$$= 2\int_0^2 e^{2x}dx - \int_0^2 xe^{2x}\,dx$$

$$= 2\int_0^2 e^{2x}dx \quad - \int_0^2 xe^{2x}\,dx$$

Doesn't require parts. we can integrate straight away

use parts

$u = x \quad\quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 1 \quad\quad v = \frac{1}{2}e^{2x}$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$= \frac{2e^{2x}}{2} - \left[x\left(\frac{1}{2}e^{2x}\right) - \int \left(\frac{1}{2}e^{2x}\right)(1)dx\right]$

Simplify:

$= e^{2x} - \left[\left(\frac{x}{2}e^{2x}\right) - \int \frac{1}{2}e^{2x}dx\right]$

Taking out the constant $\frac{1}{2}$ to make it easier:

$= e^{2x} - \left[\frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}dx\right]$

Integrating:

$= e^{2x} - \left[\frac{x}{2}e^{2x} - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right)\right]$

Simplify:

$= e^{2x} - \left(\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}\right)$

$= e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x}$

$= \frac{5}{4}e^{2x} - \frac{x}{2}e^{2x}$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$= \left[\frac{5}{4}e^{2x} - \left(\frac{x}{2}e^{2x}\right)\right]_0^2$

$= \left(\frac{5}{4}e^4 - \left(\frac{2}{2}e^4\right)\right) - \left(\frac{5}{4}e^0 - \left(\frac{0}{2}e^0\right)\right)$

$= \left(\frac{5}{4}e^4 - e^4\right) - \left(\frac{5}{4}\right)$

$= \frac{1}{4}e^4 - \frac{5}{4}$

Q. E. D.

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

2.1.2 Power and Ln

12) $\int_1^4 x^{\frac{1}{2}} \ln 2x\, dx$. Give your answer in the form $a\ln 2 + b$

$$\int_1^4 x^{\frac{1}{2}} \ln 2x\, dx$$

$u = \ln 2x$ $\quad\quad$ $\frac{dv}{dx} = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{2}{2x} = \frac{1}{x}$ $\quad\quad$ $v = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2x^{\frac{3}{2}}}{3}$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= \ln 2x \left(\frac{2x^{\frac{3}{2}}}{3}\right) - \int \left(\frac{2x^{\frac{3}{2}}}{3}\right)\left(\frac{1}{x}\right) dx$$

Simplify:

$$= \frac{2}{3}x^{\frac{3}{2}} \ln 2x - \int \frac{2x^{\frac{1}{2}}}{3} dx$$

Taking out the constant $\frac{2}{3}$ to make it easier:

$$= \frac{2}{3}x^{\frac{3}{2}} \ln 2x - \frac{2}{3}\int x^{\frac{1}{2}} dx$$

Integrating:

$$= \frac{2}{3}x^{\frac{3}{2}} \ln 2x - \frac{2}{3}\left(\frac{2}{3}x^{\frac{3}{2}}\right)$$

$$= \frac{2}{3}x^{\frac{3}{2}} \ln 2x - \frac{4}{9}x^{\frac{3}{2}}$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= \left[\frac{2}{3}\left(x^{\frac{3}{2}}\right) \ln 2x - \frac{4}{9}x^{\frac{3}{2}}\right]_1^4$$

$$= \left[\frac{2}{3}\left(4^{\frac{3}{2}}\right) \ln 8 - \frac{4}{9}(4)^{\frac{3}{2}}\right] - \left[\left(\frac{2}{3}\left(1^{\frac{3}{2}}\right) \ln 2 - \frac{4}{9}(1)^{\frac{3}{2}}\right)\right]$$

$$= \left[\frac{2}{3}(8\ln 8) - \frac{4}{9}(8)\right] - \left[\frac{2}{3}\ln 2 - \frac{4}{9}\right]$$

$$= \left(\frac{16}{3}\ln 8 - \frac{32}{9}\right) - \left(\frac{2}{3}\ln 2 - \frac{4}{9}\right)$$

$$= \frac{16}{3}\ln 8 - \frac{32}{9} - \frac{2}{3}\ln 2 + \frac{4}{9}$$

$$= \frac{16}{3}\ln 2^3 - \frac{32}{9} - \frac{2}{3}\ln 2 + \frac{4}{9}$$

$$= \frac{46}{3}\ln 2 - \frac{28}{9}$$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

13) $\int_1^3 (x-1)\ln x\,dx$. Show that the exact value is $\frac{3}{2}ln3$

$$\int_1^3 (x-1)\,ln\,x\,dx$$

$u = ln\,x$ $\quad\quad$ $\frac{dv}{dx} = x - 1$

$\downarrow$ $\quad\quad$ $\downarrow$

$\frac{du}{dx} = \frac{1}{x}$ $\quad\quad$ $v = \frac{x^2}{2} - x$

Plugging into formula: $uv - \int v\frac{du}{dx}dx$

$$= \ln x\left(\frac{x^2}{2} - x\right) - \int\left(\frac{x^2}{2} - x\right)\left(\frac{1}{x}\right)dx$$

Simplify:

$$= \left(\frac{x^2}{2} - x\right)\ln x \ - \int\left(\frac{x}{2} - 1\right)\,dx$$

Integrating:

$$= \left(\frac{x^2}{2} - x\right)\,ln\,x \ - \left(\frac{x^2}{4} - x\right)$$

Simplify:

$$= \left(\frac{x^2}{2} - x\right)\ln x \ - \frac{x^2}{4} + x$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= \left[\left(\frac{x^2}{2} - x\right)\ln x \ - \frac{x^2}{4} + x\right]_1^3$$

$$= \left(\left(\frac{3^2}{2} - 3\right)\ln 3 \ - \frac{3^2}{4} + 3\right) - \left(\left(\frac{1^2}{2} - 1\right)\ln 1 \ - \frac{1^2}{4} + 1\right)$$

$$= \left(\frac{3}{2}\ln 3 \ - \frac{9}{4} + 3\right) - \left(0 \ - \frac{1}{4} + 1\right)$$

$$= \frac{3}{2}\ln 3 - \frac{9}{4} + 3 + \frac{1}{4} - 1$$

$$= \frac{3}{2}\ln 3 + 0$$

$$= \frac{3}{2}\ln 3$$

Q. E. D.

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

14) $\int_1^8 \frac{1}{\sqrt[3]{x}} \ln x \, dx$. Give your answer in the form $a\ln 2 + b$

$$\int_1^8 \frac{1}{\sqrt[3]{x}} \ln x \, dx$$

$u = \ln x$ $\quad\quad$ $\frac{dv}{dx} = \frac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}}$

$\downarrow$ $\quad\quad$ $\downarrow$

$\frac{du}{dx} = \frac{1}{x}$ $\quad\quad$ $v = \frac{x^{\frac{2}{3}}}{\frac{2}{3}} = \frac{3}{2}x^{\frac{2}{3}}$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= \ln x \left(\frac{3}{2}x^{\frac{2}{3}}\right) - \int \left(\frac{3}{2}x^{\frac{2}{3}}\right)\left(\frac{1}{x}\right) dx$$

Simplify:

$$= \frac{3}{2}x^{\frac{2}{3}} \ln x - \int \left(\frac{3}{2}x^{\frac{2}{3}}\right)(x^{-1}) dx$$

$$= \frac{3}{2}x^{\frac{2}{3}} \ln x - \int \left(\frac{3}{2}x^{-\frac{1}{3}}\right) dx$$

Taking out the constant $\frac{3}{2}$ to make it easier:

$$= \frac{3}{2}x^{\frac{2}{3}} \ln x - \frac{3}{2}\int x^{-\frac{1}{3}} dx$$

Integrating:

$$= \frac{3}{2}x^{\frac{2}{3}} \ln x - \frac{3}{2}\left(\frac{3}{2}x^{\frac{2}{3}}\right)$$

Simplify:

$$= \frac{3}{2}x^{\frac{2}{3}} \ln x - \frac{9}{4}x^{\frac{2}{3}}$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= \left[\left(\frac{3}{2}x^{\frac{2}{3}}\right)\ln 8 - \frac{9}{4}x^{\frac{2}{3}}\right]_1^8$$

$$= \left[\left(\frac{3}{2}8^{\frac{2}{3}}\right)\ln 8 - \frac{9}{4}(8)^{\frac{2}{3}}\right] - \left[\left(\frac{3}{2}1^{\frac{2}{3}}\right)\ln 1 - \frac{9}{4}(1)^{\frac{2}{3}}\right]$$

$$= \left[\left(\frac{3}{2}(4)\right)\ln 8 - \frac{9}{4}(4)\right] - \left[0 - \frac{9}{4}\right]$$

$$= (6\ln 8 - 9) - \left(-\frac{9}{4}\right)$$

$$= 6\ln 8 - 9 + \frac{9}{4}$$

$$= 6\ln 8 - \frac{27}{4}$$

$= 6\ln 2^3 - \frac{27}{4}$

$= 6(3\ln 2) - \frac{27}{4}$

$= 18\ln 2 - \frac{27}{4}$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

15) Show that $\int_1^{e^2} x^3 \ln x \, dx = ae^8 + b$, where a and b are rational constants to be found

$$\int_1^{e^2} x^3 \ln x \, dx$$

$u = \ln x \qquad \frac{dv}{dx} = x^3$

$\downarrow \qquad \downarrow$

$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{1}{4}x^4$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$= \ln x \left(\frac{1}{4}x^4\right) - \int \left(\frac{1}{4}x^4\right)\left(\frac{1}{x}\right) dx$

Simplify:

$= \frac{1}{4}x^4 \ln x - \int \left(\frac{1}{4}x^3\right) dx$

Taking out the constant $\frac{1}{4}$ to make it easier:

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx$

Integrating:

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\left(\frac{1}{4}x^4\right)$

Simplify:

$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$= \left[\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\right]_1^{e^2}$

$$=\left[\frac{1}{4}(e^2)^4\ln e-\frac{1}{16}(e^2)^4\right]-\left[\left(\frac{1}{4}1^4\right)\ln 1-\frac{1}{16}(1)^4\right]$$
$$=\left(\frac{1}{4}e^8-\frac{1}{16}e^8\right)-\left(-\frac{1}{16}\right)$$
$$=\left(\frac{3}{16}e^8\right)--\frac{1}{16}$$
$$=\frac{3}{16}e^8+\frac{1}{16}$$
$$\therefore a=\frac{3}{16}, b=\frac{1}{16}$$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

2.1.3 Power and Power

16) $\int_{-1}^{1}5x\sqrt{2-x}dx$. Show that this equals $\frac{1}{3}(6\sqrt{3}-14)$

$$\int_{-1}^{1}5x\sqrt{2-x}dx$$

$$u=5x \qquad \frac{dv}{dx}=(2-x)^{\frac{1}{2}}$$

$$\downarrow \qquad \downarrow$$

$$\frac{du}{dx}=5 \qquad v=\frac{(2-x)^{\frac{3}{2}}}{\frac{3}{2}\times(-1)}=-\frac{2}{3}(2-x)^{\frac{3}{2}}$$

Plugging into formula: $uv-\int v\frac{du}{dx}dx$
$$=5x\left(-\frac{2}{3}(2-x)^{\frac{3}{2}}\right)-\int\left(-\frac{2}{3}(2-x)^{\frac{3}{2}}\right)(5)dx$$

Simplify:
$$=-\frac{10x}{3}(2-x)^{\frac{3}{2}}-\int-\frac{10}{3}(2-x)^{\frac{3}{2}}dx$$

Taking out the constant $-\frac{10}{3}$ **to make it easier:**
$$=-\frac{10x}{3}(2-x)^{\frac{3}{2}}+\frac{10}{3}\int(2-x)^{\frac{3}{2}}dx$$

Integrating:
$$=-\frac{10x}{3}(2-x)^{\frac{3}{2}}+\frac{10}{3}\left(\frac{(2-x)^{\frac{5}{2}}}{\frac{5}{2}\times-1}\right)$$

Simplify:
$$=-\frac{10x}{3}(2-x)^{\frac{3}{2}}+\frac{10}{3}\left(-\frac{2}{5}(2-x)^{\frac{5}{2}}\right)$$
$$=-\frac{10x}{3}(2-x)^{\frac{3}{2}}-\frac{4}{3}(2-x)^{\frac{5}{2}}$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= \left[-\frac{10x}{3}(2-x)^{\frac{3}{2}} - \frac{4}{3}(2-x)^{\frac{5}{2}}\right]_{-1}^{1}$$

$$= \left[-\frac{10}{3}(2-1)^{\frac{3}{2}} - \frac{4}{3}(2-1)^{\frac{5}{2}}\right] - \left[-\frac{10(-1)}{3}(2-(-1))^{\frac{3}{2}} - \frac{4}{3}(2-(-1))^{\frac{5}{2}}\right]$$

$$= \left[-\frac{10}{3}(1)^{\frac{3}{2}} - \frac{4}{3}(1)^{\frac{5}{2}}\right] - \left[\frac{10}{3}3^{\frac{3}{2}} - \frac{4}{3}(3)^{\frac{5}{2}}\right]$$

$$= \left(-\frac{10}{3} - \frac{4}{3}\right) - \left(\frac{10}{3}3^{\frac{3}{2}} - \frac{4}{3}(3)^{\frac{5}{2}}\right)$$

To get in the requested form we need to use rule $x^{\frac{a}{b}} = \left(\sqrt[b]{x}\right)^a$:

$$= \left(-\frac{10}{3} - \frac{4}{3}\right) - \left(\frac{10}{3}\left(\sqrt{3}\right)^3 - \frac{4}{3}\left(\sqrt{3}\right)^5\right)$$

$$= \left(-\frac{14}{3}\right) - \left(\frac{10}{3}(3\sqrt{3}) - \frac{4}{3}(9\sqrt{3})\right)$$

$$= \left(-\frac{14}{3}\right) - (10\sqrt{3} - 8\sqrt{3})$$

$$= -\frac{14}{3} - 2\sqrt{3}$$

$$= \frac{1}{3}(6\sqrt{3} - 14)$$

Q. E. D.

Note: alternatively, we could also have used a substitution here: $u = 2 - x$

17) $\int_1^5 (x-1)\sqrt{5-x}\,dx$

$$\int_1^5 (x-1)\sqrt{5-x}\,dx$$

$u = (x-1)$ $\qquad$ $\frac{dv}{dx} = (5-x)^{\frac{1}{2}}$

↓ $\qquad$ ↓

$\frac{du}{dx} = 1$ $\qquad$ $v = \frac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}\times(-1)} = -\frac{2}{3}(5-x)^{\frac{3}{2}}$

Plugging into formula: $uv - \int v\frac{du}{dx}\,dx$

$$= (x-1)\left(-\frac{2}{3}(5-x)^{\frac{3}{2}}\right) - \int\left(-\frac{2}{3}(5-x)^{\frac{3}{2}}\right)(1)\,dx$$

Simplify:

$$= -\frac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \int -\frac{2}{3}(5-x)^{\frac{3}{2}}\,dx$$

Taking out the constant $-\frac{2}{3}$ to make it easier:

$$= -\frac{2}{3}(x-1)(5-x)^{\frac{3}{2}} + \frac{2}{3}\int (5-x)^{\frac{3}{2}}\,dx$$

Integrating:

$$= -\frac{2}{3}(x-1)(5-x)^{\frac{3}{2}} + \frac{2}{3}\left(\frac{(5-x)^{\frac{5}{2}}}{\frac{5}{2}\times(-1)}\right)$$

Simplify:

$$= -\frac{2}{3}(x-1)(5-x)^{\frac{3}{2}} + \frac{2}{3}\left(-\frac{2}{5}(5-x)^{\frac{5}{2}}\right)$$

$$= -\frac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \frac{4}{15}(5-x)^{\frac{5}{2}}$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= \left[-\frac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \frac{4}{15}(5-x)^{\frac{5}{2}}\right]_1^5$$

$$= \left[-\frac{2}{3}(5-1)(5-5)^{\frac{3}{2}} - \frac{4}{15}(5-5)^{\frac{5}{2}}\right] - \left[-\frac{2}{3}(1-1)(5-1)^{\frac{3}{2}} - \frac{4}{15}(5-1)^{\frac{5}{2}}\right]$$

$$= \left(-\frac{2}{3}(4)(0)^{\frac{3}{2}} - \frac{4}{15}(0)^{\frac{5}{2}}\right) - \left(-\frac{2}{3}(0)(4)^{\frac{3}{2}} - \frac{4}{15}(4)^{\frac{5}{2}}\right)$$

$$= (0-0) - \left(0 - \frac{4}{15}(4)^{\frac{5}{2}}\right)$$

$$= \frac{4}{15}(32)$$

$$= \frac{128}{15}$$

Note: alternatively, we could also have used a substitution here: $u = 5 - x$

18) $\int_0^2 2x\sqrt{x+2}dx$. Show that this is equal to $\frac{32}{15}(2+\sqrt{2})$

$$\int_0^2 2x\sqrt{x+2}\,dx$$

$u = 2x$ $\quad$ $\frac{dv}{dx} = (2+x)^{\frac{1}{2}}$

↓ $\quad$ ↓

$\frac{du}{dx} = 2$ $\quad$ $v = \frac{(2+x)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}(2+x)^{\frac{3}{2}}$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= 2x\left(\frac{2}{3}(2+x)^{\frac{3}{2}}\right) - \int \left(\frac{2}{3}(2+x)^{\frac{3}{2}}\right)(2)\, dx$$

Simplify:

$$= \frac{4x}{3}(2+x)^{\frac{3}{2}} - \int \frac{4}{3}(2+x)^{\frac{3}{2}}\, dx$$

Taking out the constant $\frac{4}{3}$ to make it easier:

$$= \frac{4x}{3}(2+x)^{\frac{3}{2}} - \frac{4}{3}\int (2+x)^{\frac{3}{2}}\, dx$$

Integrating:

$$= \frac{4x}{3}(2+x)^{\frac{3}{2}} - \frac{4}{3}\left(\frac{(2+x)^{\frac{5}{2}}}{\frac{5}{2}}\right)$$

Simplify:

$$= \frac{4x}{3}(2+x)^{\frac{3}{2}} - \frac{4}{3}\left(\frac{2}{5}(2+x)^{\frac{5}{2}}\right)$$

$$= \frac{4x}{3}(2+x)^{\frac{3}{2}} - \frac{8}{15}(2+x)^{\frac{5}{2}}$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= \left[\frac{4x}{3}(2+x)^{\frac{3}{2}} - \frac{8}{15}(2+x)^{\frac{5}{2}}\right]_0^2$$

$$= \left[\frac{8}{3}(2+2)^{\frac{3}{2}} - \frac{8}{15}(2+2)^{\frac{5}{2}}\right] - \left[\frac{0}{3}(2+0)^{\frac{3}{2}} - \frac{8}{15}(2+0)^{\frac{5}{2}}\right]$$

$$= \left(\frac{8}{3}(4)^{\frac{3}{2}} - \frac{8}{15}(4)^{\frac{5}{2}}\right) - \left(0 - \frac{8}{15}(2)^{\frac{5}{2}}\right)$$

$$= \left(\frac{8}{3}(8) - \frac{8}{15}(32)\right) - \left(-\frac{8}{15}(\sqrt{2})^5\right)$$

$$= \left(\frac{64}{3} - \frac{256}{15}\right) - \left(-\frac{8}{15}(4\sqrt{2})\right)$$

$$= \left(\frac{64}{3} - \frac{256}{15}\right) - \left(-\frac{8}{15}(4\sqrt{2})\right)$$

$$= \frac{64}{15} + \frac{32}{15}\sqrt{2}$$

$$= \frac{32}{15}(2 + \sqrt{2})$$

Q.E.D.

Note: alternatively, we could also have used a substitution here: $u = x + 2$

2.2 Parts More Than Once (Twice)

19) $\int x^2 e^{3x} dx$

$$\int x^2 e^{3x} dx$$

$u = x^2$ $\quad \frac{dv}{dx} = e^{3x}$

$\downarrow$ $\quad \downarrow$

$\frac{du}{dx} = 2x$ $\quad v = \frac{1}{3}e^{3x}$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$= x^2\left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right)(2x)\, dx$

Simplify:

$= \frac{1}{3}x^2e^{3x} - \int \frac{2}{3}xe^{3x}\, dx$

Taking out the constant $\frac{2}{3}$ to make it easier:

$= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}\, dx$

$= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}dx$ **

In the green integral, we once again have two unrelated functions and thus we need to do integration by parts again

$u = x$ $\quad \frac{dv}{dx} = e^{3x}$

$\downarrow$ $\quad \downarrow$

$\frac{du}{dx} = 1$ $\quad v = \frac{1}{3}e^{3x}$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

Simplify:

$= x\left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right)(1)\, dx$

$= \frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x}\, dx$

Taking out the constant $\frac{1}{3}$ to make it easier:

$$= \frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x}\,dx$$

Integrating:

$$= \frac{x}{3}e^{3x} - \frac{1}{3}\left(\frac{1}{3}e^{3x}\right) + c$$

Simplify:

$$= \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + c$$

(Always remember to add the constant of integration + c)

Combining both integrals from **:

$$= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left(\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right) + c$$

Simplify:

$$= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} - \frac{2}{27}e^{3x} + c$$

20) $\int x^2 cos2x dx$

$$\int x^2 \cos 2x\,dx$$

$u = x^2$ $\quad\quad$ $\frac{dv}{dx} = \cos 2x$

↓ $\quad\quad$ ↓

$\frac{du}{dx} = 2x$ $\quad\quad$ $v = \frac{1}{2}sin\ 2x$

Plugging into formula: $uv - \int v\frac{du}{dx}dx$

$$= x^2\left(\frac{1}{2}sin\ 2x\right) - \int\left(\frac{1}{2}sin\ 2x\right)(2x)\,dx$$

Simplify:

$$= \frac{1}{2}x^2\sin 2x - \int (x\sin 2x)\,dx$$

$$= \frac{1}{2}x^2\sin 2x - \int x\sin 2x\,dx \quad **$$

In the green integral, we once again have two unrelated functions and thus we need to do integration by parts again

By parts again: $\int x\sin 2x\,dx$

$u = x$ $\quad\quad$ $\frac{dv}{dx} = \sin 2x$

↓

$$\frac{du}{dx} = 1 \qquad v = -\frac{1}{2}\cos 2x$$

Plugging into formula: $uv - \int v\frac{du}{dx}dx$

$$= x\left(-\frac{1}{2}\cos 2x\right) - \int\left(-\frac{1}{2}\cos 2x\right)(1)\,dx$$

Simplify:

$$= -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x\,dx$$

Taking out the constant $-\frac{1}{2}$ **to make it easier:**

$$= -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx$$

Integrating:

$$= -\frac{1}{2}x\cos 2x + \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) + c$$

$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$$

(Always remember to add the constant of integration + c)

Combining both integrals from **:

$$= \frac{1}{2}x^2\sin 2x - \left(-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right) + c$$

Simplify:

$$= \frac{1}{2}x^2\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + c$$

21) $\int_0^1 x^2e^x\,dx$

$$\int_0^1 x^2e^x dx$$

$$u = x^2 \qquad \frac{dv}{dx} = e^x$$

$$\frac{du}{dx} = 2x \qquad v = e^x$$

Plugging into formula: $uv - \int v\frac{du}{dx}dx$

$$= x^2(e^x) - \int (e^x)(2x)\,dx$$

Simplify:

$$= x^2e^x - \int 2xe^x \, dx$$

Taking out the constant 2 to make it easier:

$$= x^2e^x - 2\int xe^x \, dx \quad ①$$

In the green integral, we once again have two unrelated functions and thus we need to do integration by parts again

By parts again: $\int xe^x \, dx$

$u = x$ $\qquad \frac{dv}{dx} = e^x$

↓ $\qquad$ ↓

$\frac{du}{dx} = 1$ $\qquad v = e^x$

Plugging into formula: $uv - \int v\frac{du}{dx}dx$

$$= x(e^x) - \int (e^x)(1) \, dx$$

Simplify:

$$= xe^x - \int e^x \, dx$$

Integrating:

$$= xe^x - e^x$$

(Since we have limits, we don't need to add the constant of integration + c)

Combining both integrals from ①:

$$= x^2e^x - 2(xe^x - e^x)$$

Simplify:

$$= x^2e^x - 2xe^x + 2e^x$$

Applying limits:

$$= [x^2e^x - 2xe^x + 2e^x]_0^1$$

$$= (1^2e^1 - 2e^1 + 2e^1) - (0^2e^0 - 0e^0 + 2e^0)$$

$$= (e - 2e + 2e) - (0 - 0 + 2)$$

$$= (e) - (2)$$

$$= e - 2$$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

3 Gold

3.1 Parts Once

3.1.1 Power and Inverse Trig

22) $\int x^2 \tan^{-1} x \, dx$

$$\int x^2 \tan^{-1} x \, dx$$

$u = \tan^{-1} x$ $\qquad$ $\frac{dv}{dx} = x^2$

$\downarrow$ $\qquad$ $\downarrow$

$\frac{du}{dx} = \frac{1}{1+x^2}$ $\qquad$ $v = \frac{1}{3}x^3$

Plugging into formula: $uv - \int v \frac{du}{dx} \, dx$

$$= (\tan^{-1} x)\left(\frac{1}{3}x^3\right) - \int \left(\frac{1}{3}x^3\right)\left(\frac{1}{1+x^2}\right) dx$$

Simplify:

$$= \frac{1}{3}x^3 \tan^{-1} x - \int \frac{1}{3}\left(\frac{x^3}{1+x^2}\right) dx$$

Taking out the constant $\frac{1}{3}$ to make it easier:

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1+x^2} \, dx$$

$\frac{x^3}{1+x^2}$ **is top heavy, we need to divide before we can integrate (SEE POLYNOMIAL DIVISION AND INTEGRATION BASICS - FRACTION SECTION WORKSHEETS IF STRUGGLING WITH THE CONCEPT):**

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1+x^2} \, dx$$

$\longrightarrow$ **using long division, we get** $x - \frac{x}{x^2+1}$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int x - \frac{x}{x^2+1}\, dx$$

Integrating:

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\left(\frac{x^2}{2} - \frac{1}{2}\ln(x^2+1)\right) + c$$

Simplify:

$$= -\frac{1}{3}x^3 \tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6}\ln(x^2+1) + c$$

(Always remember to add the constant of integration + c)

23) $\int 2x^3 \tan^{-1} x\, dx$

$$\int 2x^3 \tan^{-1} x\, dx$$

$u = \tan^{-1} x$ $\qquad$ $\frac{dv}{dx} = 2x^3$

↓ $\qquad$ ↓

$\frac{du}{dx} = \frac{1}{1+x^2}$ $\qquad$ $v = \frac{2x^4}{4} = \frac{1}{2}x^4$

Plugging into formula: $uv - \int v\frac{du}{dx}\, dx$

$$= (\tan^{-1} x)\left(\frac{1}{2}x^4\right) - \int \left(\frac{1}{2}x^4\right)\left(\frac{1}{1+x^2}\right)\, dx$$

Simplify:

$$= \frac{1}{2}x^4 \tan^{-1} x - \int \frac{1}{2}\left(\frac{x^4}{1+x^2}\right)\, dx$$

Taking out the constant $\frac{1}{2}$ to make it easier:

$$= \frac{1}{2}x^4 \tan^{-1} x - \frac{1}{2}\int \frac{x^4}{1+x^2}\, dx$$

$\frac{x^4}{1+x^2}$ **is top heavy, we need to divide before we can integrate (SEE POLYNOMIAL DIVISION AND INTEGRATION BASICS - FRACTION SECTION WORKSHEETS IF STRUGGLING WITH THE CONCEPT):**

$$= \frac{1}{2}x^4 \tan^{-1} x - \frac{1}{2}\int \frac{x^4}{1+x^2}\, dx$$

→ **using long division, we get** $x^2 - 1 + \frac{1}{x^2+1}$

$$= \frac{1}{2}x^4 \tan^{-1} x - \frac{1}{2}\int \left(x^2 - 1 + \frac{1}{x^2+1}\right)\, dx$$

Integrating:

$$= \frac{1}{2}x^4 \tan^{-1} x - \frac{1}{2}\left(\frac{x^3}{3} - x + \tan^{-1} x\right) + c$$

Simplify:

$$= \frac{1}{2}x^4 \tan^{-1} x - \frac{1}{6}x^3 + \frac{1}{2}x - \frac{1}{2}\tan^{-1} x + c$$

(Always remember to add the constant of integration + c)

3.1.2 Power and Ln

24) $\int_0^1 x \ln(x+1) dx$

$$\int_0^1 x \ln(x+1)\, dx$$

$u = \ln(x+1)$ $\quad$ $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x+1}$ $\quad$ $v = \frac{1}{2}x^2$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= (\ln(x+1))\left(\frac{1}{2}x^2\right) - \int \left(\frac{1}{2}x^2\right)\left(\frac{1}{x+1}\right) dx$$

Simplify:

$$= \frac{1}{2}x^2 \ln(x+1) - \int \frac{1}{2}\left(\frac{x^2}{x+1}\right) dx$$

Taking out the constant $\frac{1}{2}$ to make it easier:

$$= \frac{1}{2}x^2 \ln(x+1) - \frac{1}{2}\int \left(\frac{x^2}{x+1}\right) dx$$

$\frac{x^2}{x+1}$ **is top heavy, we need to divide before we can integrate(SEE POLYNOMIAL DIVISION AND INTEGRATION BASICS - FRACTION SECTION WORKSHEETS IF STRUGGLING WITH THE CONCEPT):**

$$= \frac{1}{2}x^2 \ln(x+1) - \frac{1}{2}\int \left(\frac{x^2}{x+1}\right) dx$$

using long division, we get $x - 1 + \frac{1}{x+1}$

$$= \frac{1}{2}x^2 \ln(x+1) - \frac{1}{2}\int \left(x - 1 + \frac{1}{x+1}\right) dx$$

Integrating:

$$= \frac{1}{2}x^2 \ln(x+1) - \frac{1}{2}\left(\frac{1}{2}x^2 - x + \ln(x+1)\right)$$

Simplify:

$$= \frac{1}{2}x^2\ln(x+1) - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{2}\ln(x+1)$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= \left[\frac{1}{2}x^2\ln(x+1) - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{2}\ln(x+1)\right]_0^1$$

$$= \left(\tfrac{1}{2}\ln(1+1) - \tfrac{1}{4} + \tfrac{1}{2} - \tfrac{1}{2}\ln(1+1)\right) - \left(0 - 0 + 0 - \tfrac{1}{2}\ln(0+1)\right)$$

$$= \left(\tfrac{1}{2}\ln(2) - \tfrac{1}{4} + \tfrac{1}{2} - \tfrac{1}{2}\ln(2)\right) - \left(\tfrac{1}{2}\ln(1)\right)$$

$$= \left(-\tfrac{1}{4} + \tfrac{1}{2}\right) - (0)$$

$$= \tfrac{1}{4}$$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

25) $\int_0^{\frac{\pi}{2}} x(4cos^2 x - 3sin^2 x)\,dx$

$$\int_0^{\frac{\pi}{2}} x(4cos^2 x - 3sin^2 x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} 4xcos^2 x - 3xsin^2 x\,dx$$

$$= \int_0^{\frac{\pi}{2}} 4xcos^2 x\,dx - \int_0^{\frac{\pi}{2}} 3xsin^2 x\,dx$$

Take the constants out

$$= 4\int_0^{\frac{\pi}{2}} xcos^2 x - 3\int_0^{\frac{\pi}{2}} xsin^2 x\,dx$$

$u = x$ $\quad \frac{dv}{dx} = cos^2 x$

$\downarrow$

$\frac{du}{dx} = 1$

$u = x$ $\quad \frac{dv}{dx} = sin^2 x$

$\downarrow$

$\frac{du}{dx} = 1$

We cannot directly integrate cos^2x and sin^2x directly, thus we need to convert them use double angle formula:

$$\frac{dv}{dx} = cos^2x = \frac{1}{2}cos2x + \frac{1}{2} \qquad \frac{dv}{dx} = sin^2x = \frac{1}{2} - \frac{1}{2}cos2x$$

$$v = \frac{1}{4}sin2x + \frac{1}{2}x \qquad v = \frac{1}{2}x - \frac{1}{4}sin2x$$

Plugging into formula: $uv - \int v\frac{du}{dx}dx$

$$= 4\left[(x)\left(\frac{1}{4}sin2x + \frac{1}{2}x\right) - \int\left(\frac{1}{4}sin2x + \frac{1}{2}x\right)(1)\,dx\right] - 3\left[(x)\left(\frac{1}{2}x - \frac{1}{4}sin2x\right) - \int\left(\frac{1}{2}x - \frac{1}{4}sin2x\right)(1)\,dx\right]$$

Simplify:

$$= 4\left[\frac{1}{4}xsin2x + \frac{1}{2}x^2 - \int\left(\frac{1}{4}sin2x + \frac{1}{2}x\right)dx\right] - 3\left[\frac{1}{2}x^2 - \frac{1}{4}xsin2x - \int\left(\frac{1}{2}x - \frac{1}{4}sin2x\right)dx\right]$$

Integrating:

$$= 4\left[\frac{1}{4}xsin2x + \frac{1}{2}x^2 - \left(-\frac{1}{8}cos2x + \frac{1}{4}x^2\right)\right] - 3\left[\frac{1}{2}x^2 - \frac{1}{4}xsin2x - \left(\frac{1}{4}x^2 + \frac{1}{8}cos2x\right)\right]$$

Simplify:

$$= 4\left[\frac{1}{4}xsin2x + \frac{1}{2}x^2 + \frac{1}{8}cos2x - \frac{1}{4}x^2\right] - 3\left[\frac{1}{2}x^2 - \frac{1}{4}xsin2x - \frac{1}{4}x^2 - \frac{1}{8}cos2x\right]$$

$$= 4\left[\frac{1}{4}xsin2x + \frac{1}{4}x^2 + \frac{1}{8}cos2x\right] - 3\left[\frac{1}{4}x^2 - \frac{1}{4}xsin2x - \frac{1}{8}cos2x\right]$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= 4\left[\frac{1}{4}xsin2x + \frac{1}{4}x^2 + \frac{1}{8}cos2x\right]_0^{\frac{\pi}{2}} - 3\left[\frac{1}{4}x^2 - \frac{1}{4}xsin2x - \frac{1}{8}cos2x\right]_0^{\frac{\pi}{2}}$$

$$= 4\left[\left(\frac{1}{4}\left(\frac{\pi}{2}\right)sin\pi + \frac{1}{4}\left(\frac{\pi}{2}\right)^2 + \frac{1}{8}cos\pi\right) - \left(0 + 0 + \frac{1}{8}cos0\right)\right] - 3\left[\left(\frac{1}{4}\left(\frac{\pi}{2}\right)^2 - \frac{1}{4}\left(\frac{\pi}{2}\right)sin\pi - \frac{1}{8}cos\pi\right) - \left(0 - 0 - \frac{1}{8}cos0\right)\right]$$

$$= 4\left[\left(0 + \frac{\pi^2}{16} + \frac{1}{8}(-1)\right) - \left(\frac{1}{8}\right)\right] - 3\left[\left(\frac{\pi^2}{16} - 0 - \frac{1}{8}(-1)\right) - \left(-\frac{1}{8}\right)\right]$$

$$= 4\left(\frac{\pi^2}{16} - \frac{1}{8} - \frac{1}{8}\right) - 3\left(\frac{\pi^2}{16} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= 4\left(\frac{\pi^2}{16} - \frac{1}{4}\right) - 3\left(\frac{\pi^2}{16} + \frac{1}{4}\right)$$

$$= \frac{\pi^2}{4} - 1 - \frac{3\pi^2}{16} - \frac{3}{4}$$

$$= \frac{\pi^2}{16} - \frac{7}{4}$$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

3.2 Parts More Than Once (Three Times)

26) $\int x^3 e^x dx$

$$\int x^3 e^x dx$$

Hint: Do parts three times. The x^3 tells us we need to use parts three times to kill the power

$u = x^3$ $\quad\quad$ $\frac{dv}{dx} = e^x$

↓ $\quad\quad$ ↓

$\frac{dv}{dx} = 3x^2$ $\quad\quad$ $v = e^x$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= x^3(e^x) - \int (e^x)(3x^2)\ dx$$

Simplify:

$$= x^3e^x - \int 3x^2e^x\ dx$$

Taking out the constant 3 to make it easier:

$$= x^3e^x - 3\int x^2e^x\ dx$$

In the integral, we once again have two unrelated functions and thus we need to do integration by parts again.

By parts again: $\int x^2e^x\ dx$

$u = x^2$ $\quad\quad$ $\frac{dv}{dx} = e^x$

↓ $\quad\quad$ ↓

$\frac{du}{dx} = 2x$ $\quad\quad$ $v = e^x$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= x^2(e^x) - \int (e^x)(2x)\ dx$$

$= x^2e^x - \int 2xe^x\,dx$

In the integral, we once again have two unrelated functions and thus we need to do integration by parts again.

By parts again: $\int 2xe^x\,dx$

$u = 2x$ $\quad\quad \frac{dv}{dx} = e^x$

$\downarrow$ $\quad\quad \downarrow$

$\frac{du}{dx} = 2$ $\quad\quad v = e^x$

Plugging into formula: $uv - \int v\frac{du}{dx}\,dx$

$= 2x(e^x) - \int (e^x)(2)\,dx$

$= 2xe^x - \int 2e^x\,dx$

Integrating:

$= 2xe^x - 2e^x + c$

(Always remember to add the constant of integration + c)

Combining the 3 integrals:

$x^3e^x - 3\int x^2e^x\,dx$ became $x^3e^x - 3\left[x^2e^x - \int 2xe^x\,dx\right]$

$x^3e^x - 3\left[x^2e^x - \int 2xe^x\,dx\right]$ became $x^3e^x - 3\left[x^2e^x - (2xe^x - 2e^x)\right] + c$

So, we start from

$x^3e^x - 3[x^2e^x - (2xe^x - 2e^x)] + c$

$= x^3e^x - 3x^2e^x + 3(2xe^x - 2e^x) + c$

$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c$

3.3 "Hidden" Parts

3.3.1 Natural Log

27) $\int \ln x \, dx$

$$\int \ln x \, dx$$

We only have one function here, but this can be cleverly written as:

$$\int 1 \times \ln x \, dx$$

$$u = \ln x \qquad \frac{dv}{dx} = 1$$

$$\downarrow \qquad \downarrow$$

$$\frac{du}{dx} = \frac{1}{x} \qquad v = x$$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= \ln x \, (x) - \int (x) \left(\frac{1}{x}\right) dx$$

Simplify:

$$= x\ln x - \int 1 \, dx$$

Integrating:

$$= x\ln x - x + c$$

(Always remember to add the constant of integration + c)

28) $\int ln \frac{x}{2} dx$

$$\int ln \frac{x}{2} dx$$

We only have one function here, but this can be cleverly written as:

$$\int 1 \times ln \frac{x}{2} \, dx$$

$$u = ln \frac{x}{2} \qquad \frac{dv}{dx} = 1$$

$$\downarrow \qquad \downarrow$$

$$\frac{du}{dx} = \frac{\frac{1}{2}}{\frac{1}{2}x} = \frac{1}{x} \qquad v = x$$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= \ln\frac{x}{2}(x) - \int (x)\left(\frac{1}{x}\right) dx$$

Simplify:

$$= x\ln\frac{x}{2} - \int 1 \ dx$$

Integrating:

$$= x\ln\frac{x}{2} - x + c$$

(Always remember to add the constant of integration + c)

29) $\int \ln 2x \, dx$

$$\int \ln 2x \, dx$$

We only have one function here, but this can be cleverly written as:

$$\int 1 \times \ln 2x \ dx$$

$u = \ln 2x$ $\quad\quad$ $\frac{dv}{dx} = 1$

$\downarrow$ $\quad\quad$ $\downarrow$

$\frac{du}{dx} = \frac{2}{2x} = \frac{1}{x}$ $\quad\quad$ $v = x$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= \ln 2x\,(x) - \int (x)\left(\frac{1}{x}\right) dx$$

Simplify:

$$= x\ln 2x - \int 1 \ dx$$

Integrating:

$$= x\ln 2x - x + c$$

(Always remember to add the constant of integration + c)

30) $\int_1^2 \ln 4x\, dx$

$$\int_1^2 \ln 4x\, dx$$

We only have one function here, but this can be cleverly written as:

$$\int 1 \times \ln 4x \; dx$$

$u = \ln 4x$ $\quad\quad$ $\frac{dv}{dx} = 1$

$\downarrow$ $\quad\quad$ $\downarrow$

$\frac{du}{dx} = \frac{4}{4x} = \frac{1}{x}$ $\quad\quad$ $v = x$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$= \ln 4x\,(x) - \int (x)(\frac{1}{x})\; dx$

Simplify:

$= x \ln 4x - \int 1 \; dx$

Integrating:

$= x \ln 4x - x$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$= \; [x \ln 4x - x]_1^2$

$= (2 \ln 8 - 2) - (\ln 4 - 1)$

$= 2 \ln 8 - 2 - \ln 4 + 1$

$= 2 \ln 8 - 1 - \ln 4$

$= 2 \ln 2^3 - 1 - \ln 2^2$

$= 2(3 \ln 2) - 1 - (2 \ln 2)$

$= 6 \ln 2 - 1 - 2 \ln 2$

$= 4 \ln 2 - 1$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

3.3.2 Inverse Trig

31) $\int \arctan x \, dx$

$$\int \arctan x \, dx$$

We only have one function here, but this can be cleverly written as:

$$\int 1 \times \arctan x \ dx$$

$$u = \arctan x \qquad \frac{dv}{dx} = 1$$

$$\downarrow \qquad \downarrow$$

$$\frac{du}{dx} = \frac{1}{1+x^2} \qquad v = x$$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= \arctan x \,(x) \ - \int (x)\left(\frac{1}{1+x^2}\right) \ dx$$

Simplify:

$$= x \arctan x \ - \int \frac{x}{1+x^2} dx$$

Integrating:

$$= x \ln 2x \ - \frac{1}{2} ln(1 + x^2) + c$$

(Always remember to add the constant of integration + c)

32) $\int \arctan 3x \, dx$

$$\int \arctan 3x \, dx$$

We only have one function here, but this can be cleverly written as:

$$\int 1 \times \arctan 3x \ dx$$

$$u = \arctan 3x \qquad \frac{dv}{dx} = 1$$

$$\downarrow \qquad \downarrow$$

$$\frac{du}{dx} = \frac{3}{1+(3x)^2} \qquad v = x$$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$= \arctan 3x\,(x) - \int (x)\left(\frac{3}{1+(3x)^2}\right) dx$

Simplify:

$= x \arctan 3x - \int \frac{3x}{1+9x^2} dx$

Taking out the constant 3 to make it easier:

$= x \arctan 3x - 3\int \frac{x}{1+9x^2} dx$

Integrating:

$= x \arctan 3x - 3\left(\frac{\ln(1+9x^2)}{18}\right) + c$

Simplify:

$= x \arctan 3x - \frac{1}{6}\ln(1+9x^2) + c$

(Always remember to add the constant of integration + c)

33) $\int_0^{0.5} \arcsin x \, dx$

$$\int_0^{0.5} \arcsin x \, dx$$

We only have one function here, but this can be cleverly written as:

$$\int 1 \times \arcsin x \, dx$$

$u = \arcsin x$ $\quad$ $\frac{dv}{dx} = 1$

↓ $\quad$ ↓

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ $\quad$ $v = x$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$= \arcsin x\,(x) - \int (x)\left(\frac{1}{\sqrt{1-x^2}}\right) dx$

Simplify:

$= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$

$= x \arcsin x - \int x\,(1-x^2)^{-0.5} dx$

Integrating:

$$= x\, arcsinx \; - \left(\frac{(1-x^2)^{0.5}}{-0.5\,(2)}\right)$$

Simplify:

$$= x\, arcsinx - (-(1-x^2)^{0.5})$$

$$= x\, arcsinx + (1-x^2)^{0.5}$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= \; [x\, arcsinx + (1-x^2)^{0.5}\,]_0^{0.5}$$

$$= (0.5\, arcsin0.5 + (1-0.5^2)^{0.5}) - (0 + (1-0)^{0.5})$$

$$= \left(0.5\left(\frac{\pi}{6}\right) + (1-0.25)^{0.5}\right) - (1^{0.5})$$

$$= \left(\frac{\pi}{12} + (0.75)^{0.5}\right) - (1)$$

$$= \left(\frac{\pi}{12} + (0.75)^{0.5}\right) - (1)$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

34) $\int_0^1 arccos\frac{x}{2}\, dx$. Show that this is $\frac{\pi}{3} + 2 - \sqrt{3}$

$$\int_0^1 \text{arccos}\frac{x}{2}\, dx$$

We only have one function here, but this can be cleverly written as:

$$\int 1 \times arccos\frac{x}{2}\; dx$$

$u = arccos\frac{x}{2}$ → $\frac{du}{dx} = -\frac{0.5}{\sqrt{1-\left(\frac{x}{2}\right)^2}}$

$\frac{dx}{dx} = 1$ → $v = x$

$$= -\frac{0.5}{\sqrt{1-\frac{x^2}{4}}}$$

$$= -\frac{0.5}{\sqrt{\frac{4-x^2}{4}}}$$

$$= -\frac{0.5}{0.5\sqrt{4-x^2}}$$

$$= -\frac{1}{\sqrt{4-x^2}}$$

Plugging into formula: $uv - \int v\frac{du}{dx}dx$

$$= arccos\frac{x}{2}(x) - \int (x)\left(-\frac{1}{\sqrt{4-x^2}}\right)\ dx$$

Simplify:

$$= x\ arccos\frac{x}{2} - \int -\frac{x}{\sqrt{4-x^2}}\ dx$$

$$= x\ arccos\frac{x}{2} - \int -x\,(4-\ x^2)^{-0.5}\ dx$$

$$= x\ arccos\frac{x}{2} + \int x\,(4-\ x^2)^{-0.5}\ dx$$

Integrating:

$$= x\ arccos\frac{x}{2} + \left(\frac{(4-x^2)^{0.5}}{0.5\times -2}\right)$$

Simplify:

$$= x\ arccos\frac{x}{2} + (-(4-\ x^2)^{0.5})$$

$$= x\ arccos\frac{x}{2} - (4-\ x^2)^{0.5}$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$= \left[x\ arccos\frac{x}{2} - (4-\ x^2)^{0.5}\right]_0^1$$

$$= \left(arccos\frac{1}{2} - (4-\ 1^2)^{0.5}\right) - (0\ arccos\,0\ - (4-0)^{0.5})$$

$$= \left(\frac{\pi}{3} - (3)^{0.5}\right) - (-(4)^{0.5})$$

$$= \left(\frac{\pi}{3} - \sqrt{3}\right) - (-2)$$

$$= \frac{\pi}{3} - \sqrt{3} + 2$$

Q.E.D.

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

4 Diamond

4.1 Parts & Substitution Combined

35) $\int_0^{0.5} x\,ln(x^2+1)dx$

$$\int_0^{0.5} x\,ln(x^2+1)dx$$

We first need to integrate using substitution:

$let\ u = x^2+1$

$\frac{du}{dx} = 2x$ upper limit, $x = 0.5$: $u = 0.5^2 + 1 = 1.25$

re-arranging

$dx = \frac{du}{2x}$ lower limit, $x = 0$: $u = 0^2 + 1 = 1$

$$\int_1^{1.25} x\,ln(u)\frac{du}{2x}$$

$$= \int_1^{1.25} \frac{1}{2} ln(u)\,du$$

$$= \frac{1}{2}\int_1^{1.25} ln(u)\,du$$

Integrating using parts now:

$$\frac{1}{2}\int_1^{1.25} 1 \times \ln u\, du$$

$u = \ln u$ $\qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{u}$ $\qquad v = u$

Plugging into formula: $uv - \int v \frac{du}{dx} du$

$= \frac{1}{2}\left[((\ln u))(u) - \int (u)\left(\frac{1}{u}\right) du\right]$

Simplify:

$= \frac{1}{2}(u \ln u - \int 1 \; du)$

Integrating:

$= \frac{1}{2}(u \ln u - u)$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$= \frac{1}{2}[u \ln u - u]_1^{1.25}$

$= \frac{1}{2}(1.25 \ln 1.25 - 1.25) - \frac{1}{2}(1 \ln 1 - 1)$

$= \frac{1}{2}(1.25 \ln 1.25 - 1.25) - \frac{1}{2}(-1)$

$= \frac{5}{8} \ln 1.25 - \frac{5}{8} + \frac{1}{2}$

$= \frac{5}{8} \ln 1.25 - \frac{1}{8}$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

36) $\int_1^4 e^{\sqrt{x}} dx$

$$\int_1^4 e^{\sqrt{x}} dx$$

We first need to integrate using substitution:

$let \; u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2} x^{-0.5} = \frac{1}{2\sqrt{x}}$

re-arranging

$dx = 2\sqrt{x} = 2u \; du$

upper limit, $x = 4$: $u = \sqrt{4} = 2$

lower limit, $x = 0$: $u = \sqrt{1} = 1$

$$\int_1^2 e^u 2u \; du$$

$$= 2\int_1^2 e^u u \; du$$

Integrating using parts now:

$$2\int_1^2 e^u u\, du$$

$u = u$ $\frac{dv}{dx} = e^u$

↓ ↓

$\frac{du}{dx} = 1$ $v = e^u$

Plugging into formula: $uv - \int v \frac{du}{dx} du$

$=2[\,2\,u\,(\,e^u) \;-\int(e^u)(1)\;du]$

Simplify:

$= 2(ue^u - \int e^u\, du)$

Integrating:

$= 2(ue^u - (e^u))$

$= 2ue^u - 2e^u$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$= \;[2ue^u - 2e^u]_1^2$

$= (2(2)e^2 - 2e^2) - (2e^1 - 2e^1)$

$= (4e^2 - 2e^2) - (2e - 2e)$

$= 2e^2$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

37) $\int_0^5 e^{\sqrt{3x+1}}\,dx\;(u = \sqrt{3x+1})$

$$\int_0^5 e^{\sqrt{3x+1}} dx$$

We first need to integrate using substitution:

$let\; u = (3x+1)^{0.5}$

$\frac{du}{dx} = 3\;\times 0.5(3x+1)^{-0.5}$ upper limit, $x = 5$: $u = (15+1)^{0.5} = 4$

$= \frac{3}{2}(3x+1)^{-0.5} = \frac{3}{2(3x+1)^{0.5}}$

lower limit, $x = 0$: $u = (0+1)^{0.5} = 1$

re-arranging ↓

$dx = \frac{2}{3}(3x+1)^{0.5} = \frac{2}{3}u\,du$

$$\int_1^4 e^u \frac{2}{3} u\,du$$

$$= \frac{2}{3}\int_1^4 e^u u\,du$$

Integrating using parts:

$$\frac{2}{3}\int_1^4 e^u u\,du$$

$u = u$ → $\frac{du}{dx} = 1$

$\frac{dv}{dx} = e^u$ → $v = e^u$

Plugging into formula: $uv - \int v\frac{du}{dx}du$

$= \frac{2}{3}(u(e^u) - \int (e^u)(1)\,du)$

Simplify:

$= \frac{2}{3}(ue^u - \int e^u\,du)$

Integrating:

$= \frac{2}{3}(ue^u - (e^u))$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$= \frac{2}{3}[ue^u - e^u]_1^4$

$= \frac{2}{3}\{(4e^4 - e^4) - (e^1 - e^1)\}$

$= \frac{2}{3}\{4e^4 - e^4 - e + e\}$

$=\frac{2}{3}\{3e^4\}$

$=2e^4$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

38) $\int_0^{\sqrt{2}} x^3 \ln(x^2+2)\,dx \ \ (u = x^2+2)$

$$\int_0^{\sqrt{2}} x^3\, ln(x^2+2)\,dx$$

We first need to integrate using substitution:

$let\ u = x^2+2 \rightarrow \quad x^2 = u-2$

$\frac{du}{dx} = 2x$

upper limit, $x=5$: $u = (\sqrt{2})^2+2 = 4$

re-arranging

$dx = \frac{1}{2x}\,du$

lower limit, $x=0$: $u = (0)^2+2 = 2$

$$\int_2^4 x^3\, ln(u)\frac{1}{2x}\,du$$

$$= \int_2^4 x^2\, ln(u)\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_2^4 (u-2)\, ln(u)\ \ du$$

Integrating using parts:

$$\frac{1}{2}\int_2^4 (u-2)\, ln\, u\ \ du$$

$u = \ln u \qquad \frac{dv}{dx} = (u-2)$

$\frac{du}{dx} = \frac{1}{u} \qquad v = \frac{1}{2}u^2 - 2u$

Plugging into formula: $uv - \int v\frac{du}{dx}du$

$=\frac{1}{2}\left[(\, ln\, u\left(\frac{1}{2}u^2-2u\right) - \int\left(\frac{1}{2}u^2-2u\right)\left(\frac{1}{u}\right)\ du\right]$

Simplify:

$$=\frac{1}{2}\left[\ \left(\frac{1}{2}u^2-2u\right)\ln u\ -\int\left(\frac{1}{2}u-2\right)\ du\right]$$

Integrating:

$$=\frac{1}{2}\left[\left(\frac{1}{2}u^2-2u\right)\ln u\ -\left(\frac{1}{4}u^2-2u\right)\right]$$

(Since we have limits, we don't need to add the constant of integration + c)

Applying limits:

$$=\ \frac{1}{2}\left[\left(\frac{1}{2}u^2-2u\right)\ln u-\left(\frac{1}{4}u^2-2u\right)\right]_2^4$$

$$=\frac{1}{2}\left\{\left[\left(\frac{1}{2}(4)^2-8\right)\ln 4-\left(\frac{1}{4}(4)^2-2(4)\right)\right]-\left[\left(\frac{1}{2}(2)^2-4\right)\ln 2-\left(\frac{1}{4}(2)^2-2(2)\right)\right]\right\}$$

$$=\frac{1}{2}\{[(8-8)\ln 4-\ (4-8)]-\ [(2-4)\ln 2-\ (1-4)]\}$$

$$=\frac{1}{2}\{[0-\ (-4)]-\ [(-2)\ln 2-\ (-3)]\}$$

$$=\frac{1}{2}\{4-[\,-2\ln 2+3]\}$$

$$=\frac{1}{2}(4+\ 2\ln 2-3)$$

$$=\frac{1}{2}(1+\ 2\ln 2)$$

$$=\frac{1}{2}+\ \ln 2$$

Note: alternatively, we could have applied the limits as we were going along, but it is easier to add the limits at the very end after all the integration is complete

39) $\int xe^{\sqrt{x}}\,dx$

$$\int x\,e^{\sqrt{x}}\ dx$$

We first need to integrate using substitution:

$$let\ u=\sqrt{x}$$

$$\frac{du}{dx}=\frac{1}{2}x^{-0.5}=\frac{1}{2\sqrt{x}} \qquad\qquad x=u^2$$

re-arranging ↓

$$dx=2\sqrt{x}=2u\ du$$

$$\int u^2\,e^u\ \ 2u\ du$$

$$=2\int\ e^u\,u^3\ du$$

Integrating using parts now:

$$2\int u^3 e^u dx$$

Hint: Do parts three times. The u^3 tells us we need to use parts three times to kill the power

$u = u^3$ $\quad\quad$ $\frac{dv}{dx} = e^u$

↓ $\quad\quad$ ↓

$\frac{du}{dx} = 3u^2$ $\quad\quad$ $v = e^u$

Plugging into formula: $uv - \int v \frac{du}{dx} du$

$= 2\left[(u^3(e^u) - \int (e^u)(3u^2)\ du\right]$

Simplify:

$= 2(u^3 e^u - \int 3u^2 e^u\ du)$

Taking out the constant 3 to make it easier to integrate:

$$= u^3 e^u - 3\int u^2 e^u\ du$$

In the integral, we once again have two unrelated functions and thus we need to do integration by parts again.

By parts again: $\int u^2 e^u\ du$

$u = u^2$ $\quad\quad$ $\frac{dv}{dx} = e^u$

↓ $\quad\quad$ ↓

$\frac{du}{dx} = 2u$ $\quad\quad$ $v = e^u$

Plugging into formula: $uv - \int v \frac{du}{dx} du$

$= u^2(e^u) - \int (e^u)(2u)\ du$

$= u^2 e^u - \int 2ue^u\ du$

In the integral, we once again have two unrelated functions and thus we need to do integration by parts again.

By parts again: $\int 2ue^u \, du$

$u = 2u \qquad \frac{dv}{dx} = e^u$

$\downarrow \qquad \downarrow$

$\frac{du}{dx} = 2 \qquad v = e^u$

Plugging into formula: $uv - \int v \frac{du}{dx} du$

$= 2u(e^u) - \int (e^u)(2) \, du$

$= 2ue^u - \int 2e^u \, du$

Integrating:

$= 2ue^u - 2e^u + c$

(Always remember to add the constant of integration + c)

Combining the 3 integrals:

$= 2\{x^3e^x - 3[x^2e^x - (2xe^x - 2e^x)]\} + c$

$= 2[x^3e^x - 3x^2e^x + 3(2xe^x - 2e^x)] + c$

$= 2(e^x - 3x^2e^x + 6xe^x - 6e^x) + c$

$= 2x^3e^x - 6x^2e^x + 12xe^x - 12e^x + c$

4.2 Cyclic/Infinite Parts

40) $\int e^x sinx \, dx$

$$\int e^x \sin x \, dx$$

$u = \sin x \qquad \frac{dv}{dx} = e^x$

$\downarrow \qquad \downarrow$

$\frac{du}{dx} = \cos x \qquad v = e^x$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$= (\sin x)(e^x) - \int (e^x)(\cos x)\, dx$

Simplify:

$= e^x \sin x - \int e^x \cos x \; dx$

In the integral, we once again have two unrelated functions and thus we need to do integration by parts again.

By parts again: $\int e^x \cos x \; dx$

$u = \cos x$ $\quad$ $\frac{dv}{dx} = e^x$

↓ $\quad$ ↓

$\frac{du}{dx} = -\sin x$ $\quad$ $v = e^x$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$= \cos x\,(e^x) - \int (e^x)(-\sin x)\, dx$

$= e^x \cos x + \int e^x \sin x \; dx$

Combining both integrals:

$= \; e^x \sin x - (e^x \cos x + \int e^x \sin x \; dx)$

$= \; e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

It appears that we need to do parts again, but if we look at this we can see that the integral we have to do is the <u>same</u> as the integral we started with in the question:

$\int e^x \sin x \, dx \; = \; e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

Let's call the integral we want I:

$I \; = \; e^x \sin x - e^x \cos x - I$

Let's now make I the subject

$2I \; = \; e^x \sin x - e^x \cos x$

$I \; = \; \frac{1}{2}(e^x \sin x - e^x \cos x) + c$

(Always remember to add the constant of integration + c)

41) $\int e^x cos3xdx$

$$\int e^x \cos 3x\, dx$$

$u = \cos 3x$ $\quad\quad$ $\frac{dv}{dx} = e^x$

↓ $\quad\quad$ ↓

$\frac{du}{dx} = -3\sin 3x$ $\quad\quad$ $v = e^x$

Plugging into formula: $uv - \int v\frac{du}{dx}dx$

$= (\cos 3x)(e^x) - \int (e^x)(-3\sin 3x)\, dx$

Simplify:

$= e^x \cos 3x - \int -3e^x \sin 3x\, dx$

$= e^x \cos 3x + 3\int e^x \sin 3x\, dx$

In the integral, we once again have two unrelated functions and thus we need to do integration by parts again.

By parts again: $\int e^x \sin 3x\, dx$

$u = \sin 3x$ $\quad\quad$ $\frac{dv}{dx} = e^x$

↓ $\quad\quad$ ↓

$\frac{du}{dx} = 3\cos 3x$ $\quad\quad$ $v = e^x$

Plugging into formula: $uv - \int v\frac{du}{dx}dx$

$= \sin 3x\,(e^x) - \int (e^x)(3\cos 3x)\, dx$

$= e^x \sin 3x - 3\int e^x \cos 3x\ dx$

Combining both integrals:

$= e^x \cos 3x + 3(e^x \sin 3x - 3\int e^x \cos 3x\ dx)$

$= e^x \cos 3x + 3e^x \sin 3x - 9\int e^x \cos 3x\ dx$

It appears that we need to do parts again, but if we look at this we can see that the integral we have to do is the <u>same</u> as the integral we started with in the question:

$\int e^x \cos 3x\, dx = e^x \cos 3x + 3e^x \sin 3x - 9\int e^x \cos 3x\, dx$

Let's call the integral we want I:

$I = e^x \cos 3x + 3e^x \sin 3x - 9I$

Let's now make I the subject

$10I = e^x \cos 3x + 3e^x \sin 3x$

$I = \frac{1}{10}(e^x \cos 3x + 3e^x \sin 3x) + c$

(Always remember to add the constant of integration + c)

42) $\int e^{3x} \sin 2x dx$

$$\int e^{3x} \sin 2x\, dx$$

$u = \sin 2x \qquad \frac{dv}{dx} = e^{3x}$

$\downarrow \qquad \downarrow$

$\frac{du}{dx} = 2\cos 2x \qquad v = \frac{1}{3}e^{3x}$

Plugging into formula: $uv - \int v\frac{du}{dx}dx$

$= (\sin 2x)(\frac{1}{3}e^{3x}) - \int (\frac{1}{3}e^{3x})(2\cos 2x)\, dx$

Simplify:

$= \frac{1}{3}e^{3x} \sin 2x - \int \frac{2}{3}e^{3x} \cos 2x\, dx$

$= \frac{1}{3}e^{3x} \sin 2x - \frac{2}{3}\int e^{3x} \cos 2x\, dx$

In the integral, we once again have two unrelated functions and thus we need to do integration by parts again.

By parts again: $\int e^{3x} \cos 2x\, dx$

$u = \cos 2x \qquad \frac{dv}{dx} = e^{3x}$

$\downarrow \qquad \downarrow$

$\frac{du}{dx} = -2\sin 2x \qquad v = \frac{1}{3}e^{3x}$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= \cos 2x \left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right)(-2\sin 2x)\, dx$$

$$= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\int e^{3x} \sin 2x \; dx$$

Combining both integrals:

$$= \frac{1}{3}e^{3x} \sin 2x - \frac{2}{3}\left(\frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\int e^{3x} \sin 2x \; dx\right)$$

$$= \frac{1}{3}e^{3x} \sin 2x - \frac{2}{9}e^{x}\cos 2x - \frac{4}{3}\int e^{3x} \sin 2x \; dx$$

It appears that we need to do parts again, but if we look at this we can see that the integral we have to do is the same as the integral we started with.

Let's call the integral we want I:

$$I = \frac{1}{3}e^{3x} \sin 2x - \frac{2}{9}e^{x}\cos 2x - \frac{4}{9}I$$

$$\frac{13}{9}I = \frac{1}{3}e^{3x} \sin 2x - \frac{2}{9}e^{x}\cos 2x$$

$$I = \frac{9}{13}\left(\frac{1}{3}e^{x} \sin 2x - \frac{2}{9}e^{x}\cos 2x\right) + c$$

$$= \frac{e^x}{13}(3\sin 2x - 2\cos 2x) + c$$

(Always remember to add the constant of integration + c)

43) $\int \frac{1}{x}\ln x \, dx$

$$\int \frac{1}{x} \ln x \, dx$$

$u = \ln x$ $\qquad$ $\frac{dv}{dx} = \frac{1}{x}$

↓ $\qquad$ ↓

$\frac{du}{dx} = \frac{1}{x}$ $\qquad$ $v = \ln x$

Plugging into formula: $uv - \int v \frac{du}{dx} dx$

$$= (\ln x)(\ln x) - \int (\ln x)\left(\frac{1}{x}\right) dx$$

Simplify:

$$= (\ln x)^2 - \int \frac{1}{x} \ln x \, dx$$

It appears that we need to do parts again, but if we look at this we can see that the integral we have to do is the same as the integral we started with.

Let's call the integral we want I:

$$I = (\ln x)^2 - I$$

$$2I = (\ln x)^2$$

$$I = \frac{1}{2}(\ln x)^2 + c$$

(Always remember to add the constant of integration + c)

Note: Could also have used recognition/reverse chain rule for this question which would have been a lot easier